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The Effect of Gravitational Waves from a Spinning Rod on Its Rigidity

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Abstract

The energy and angular momentum flux carried by gravitational waves from a spinning rod are calculated exactly in the weak field limit of general relativity. It is shown that contrary to common belief, the energy and angular momentum flux are not proportional and the energy and angular momentum equations governing the evolution of the rod are not identical. The spinning rod does not remain rigid: Its length increases. Both the angular deceleration and the rate of change of length are dependent on the nature of the material of the rod, and these rates are small, as expected.

1. *Introduction*

In the theory of general relativity, it is believed that systems of moving masses emit gravitational radiation carrying away energy and angular momentum from the systems. In the linearized version of general relativity, expressions for the rates of energy and angular momentum transported by the gravitational waves are well known (Landau and Lifschitz, 1962; Peters, 1964). In particular, the energy loss by a spinning rod was worked out by Eddington (1922) in the weak field limit with the tacit assumptions that both the length and angular velocity of the rod remain unchanged over a period.

In this paper, we shall remove the assumptions mentioned above for the case of a spinning rod in order to examine the implications when both energy as well as angular momentum losses are taken into account. In section 2, exact expressions for the energy and angular momentum transported by the gravitational waves within the weak field limit are given. We will see in section 3 that the resultant equations for the loss of energy and angular momentum are, contrary to common belief, not identical. Section 4 shows that if the rod is rigid, in general, to each value of the angular velocity there are two additional possible values for the angular deceleration—an untenable

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situation. This shows that the assumption that the rod remains rigid is inconsistent with the linearized version of general relativity. In fact, as we will see in section 5, a spinning rod emitting gravitational radiation will not only slow down, but its length will increase, the rates being dependent also on the substance of the rod. These effects are small: The rod will take an infinitely long time to slow down to zero angular velocity and the maximum fractional increase in the length is small.

2. *Energy and Angular Momentum Flux by Gravitational Radiation*

The energy and angular momentum flux due to the emission of gravitational radiation by any system is given in the weak field limit by the usual expressions (Landau and Lifschitz, 1962; Peters, 1964):

$$P_E = (G/45c^5)\langle(\dddot{\mathscr{D}}_{\alpha\beta})^2\rangle \tag{2.1}$$

$$P_{J_\alpha} = (2G/45c^5)\,\epsilon_{\alpha\beta\gamma}\,\langle\ddot{\mathscr{D}}_{\beta\delta}\dddot{\mathscr{D}}_{\gamma\delta}\rangle \tag{2.2}$$

Here

$$\mathscr{D}_{\alpha\beta} = \int \rho(r)(3x_\alpha x_\beta - \delta_{\alpha\beta}r^2)dV \tag{2.3}$$

which is the quadrupole moment tensor, while $\epsilon_{\alpha\beta\gamma}$ is a completely antisymmetric unit pseudotensor.

The uniform rod, of mass M and length $2L$, is spinning about its center of mass in the X-Y plane with an angular velocity ω about the Z axis. We find

$$\mathscr{D}_{xx} = I(3\cos^2\theta - 1) \tag{2.4}$$

$$\mathscr{D}_{xy} = \mathscr{D}_{yx} = 3I\sin\theta\cos\theta \tag{2.5}$$

$$\mathscr{D}_{yy} = I(3\sin^2\theta - 1) \tag{2.6}$$

$$\mathscr{D}_{zz} = -I \tag{2.7}$$

where

$$I = \tfrac{1}{3}ML^2 \tag{2.8}$$

which is the moment of inertia about the Z axis. Taking into account the possible changes of the length and angular velocity of the rod, we get from the above equations the following exact expressions for the energy and angular momentum flux:

$$P_E = \mathscr{E}(\gamma_1{}^2 + \gamma_2{}^2 + \gamma_3{}^2) \tag{2.9}$$

$$P_{J_z} = (\mathscr{E}/\omega)(\gamma_1\gamma_4 + \gamma_2\gamma_5) \tag{2.10}$$

where

$$\gamma_1 = 1 - \tfrac{1}{4}(\beta_2 + 6\beta_1\beta_3 + 6\beta_3{}^2 + 6\beta_4) \tag{2.11}$$

$$\gamma_2 = \tfrac{1}{4}(6\beta_1 + 12\beta_3 - 3\beta_3\beta_4 - \beta_5) \tag{2.12}$$

$$\gamma_3 = (\sqrt{3}/12)(3\beta_3\beta_4 + \beta_5) \tag{2.13}$$

$$\gamma_4 = 1 - \tfrac{1}{2}(\beta_3{}^2 + \beta_4) \tag{2.14}$$

$$\gamma_5 = \tfrac{1}{2}(\beta_1 + 4\beta_3) \tag{2.15}$$

$$\beta_1 = \dot{\omega}/\omega^2 \tag{2.16}$$

$$\beta_2 = \ddot{\omega}/\omega^3 \tag{2.17}$$

$$\beta_3 = \dot{L}/L\omega \tag{2.18}$$

$$\beta_4 = \ddot{L}/L\omega^2 \tag{2.19}$$

$$\beta_5 = \dddot{L}/L\omega^3 \tag{2.20}$$

$$\mathscr{E} = 32GI^2\omega^6/5c^5 \tag{2.21}$$

The expressions (2.9) and (2.10) for the energy and angular momentum flux reduce respectively to the usual results:

$$(P_E)_{\text{usual}} = \mathscr{E} \tag{2.22}$$

$$(P_{J_z})_{\text{usual}} = \mathscr{E}/\omega \tag{2.23}$$

when the dimensionless quantities, β_i, are all neglected. Since in general these quantities are not zero, the energy flux is not proportional to the angular momentum flux as commonly believed. The corrections to the usual expressions are small, as expected.

3. *Energy and Angular Momentum Equations*

As we are considering the motion of a nonrigid rod, we must take into account the internal potential and thermal energy of the rod. To simplify, we shall assume the rod to be at a constant temperature and thus we need only consider its internal potential energy. If the interparticle force is conservative and obeys Newton's third law in the strong form (Goldstein, 1950), an internal potential energy U can be defined. Thus the total energy of the rod is

$$E = \tfrac{1}{2}I\omega^2 + U \tag{3.1}$$

while the angular momentum is

$$J_z = I\omega \tag{3.2}$$

where, for simplicity, we have used Newtonian mechanics only (see section 6 for refinements). Taking into account the change of the length of the rod and its consequent change of internal potential energy, the above equations lead to

$$dE/dt = I\omega^3(\beta_1 + \beta_3 + \zeta) \tag{3.3}$$

and

$$dJ_z/dt = I\omega^2(\beta_1 + 2\beta_3) \tag{3.4}$$

where

$$\zeta = \dot{U}/I\omega^3 \tag{3.5}$$

This can be written as

$$\zeta = b\beta_3 \tag{3.6}$$

with

$$b = \frac{L}{I\omega^2}\frac{dU}{dL} \tag{3.7}$$

Another form for equation (3.7) is

$$b = \frac{3Y}{\rho}\frac{\Delta L}{L^3\omega^2} = \frac{3Y}{2T}\frac{\Delta L}{L}\left(\frac{\omega_M}{\omega}\right)^2 \tag{3.8}$$

where Y is the Young's modulus, ρ the density, T the tensile strength, and ω_M the maximum angular velocity beyond which the rod will break. In this form, it is clear that the quantity b is small. For example, for an aluminium rod of length 2 meters and $\omega = 100$, $b \simeq 82\ \Delta L$, which is less than 1 as ΔL is clearly small. In fact as $\omega \to 0$, $b \to 0$, a result clear from equation (5.6).

Since the rate of energy and angular momentum carried away by gravitational waves must be equal to the rate of loss of energy and angular momentum of the system respectively, we get

$$\beta_1 + \beta_3(1 + b) = -\eta[\gamma_1^2 + \gamma_2^2 + \gamma_3^2] \tag{3.9}$$

$$\beta_1 + 2\beta_3 = -\eta[\gamma_1\gamma_4 + \gamma_2\gamma_5] \tag{3.10}$$

with

$$\eta = \mathscr{E}/I\omega^3 = 32GI\omega^3/5c^5 \tag{3.11}$$

where we have used equations (2.9), (2.10), (3.3), (3.4), and (3.6). Thus, the energy equation (3.9) and angular momentum equation (3.10) are not identical in general.

4. *Gravitational Radiation from a Rigid Rod*

In this section, we assume that the rod is rigid. This implies that

$$\beta_3 = \beta_4 = \beta_5 = \zeta = 0 \tag{4.1}$$

The energy and angular momentum equations become [from equations (3.9) and (3.10)]

$$\beta_1 = -\eta[(1 - \tfrac{1}{4}\beta_2)^2 + \tfrac{9}{4}\beta_1^2] \tag{4.2}$$

$$\beta_1 = -\eta[1 - \tfrac{1}{4}\beta_2 + \tfrac{3}{4}\beta_1^2] \tag{4.3}$$

Note that the usual treatment of ignoring the length and angular velocity variations can be obtained by putting $\beta_1 = \beta_2 = 0$ on the right-hand side of equations (4.2) and (4.3). Thus the usual result is

$$(\beta_1)_{\text{usual}} = -\eta \tag{4.4}$$

For consistency, we cannot ignore β_1 and β_2 even though they are small. From equations (4.2) and (4.3), we get

$$\beta_2 = 6\beta_1^2 + \tfrac{1}{4}\beta_2^2 \tag{4.5}$$

By defining

$$y = \dot{\omega}^2 \tag{4.6}$$

$$v = \omega^4 \tag{4.7}$$

and

$$q = dy/dv \tag{4.8}$$

equation (4.5) becomes a first-order differential equation:

$$q^2 - 2q + 6y/v = 0 \tag{4.9}$$

The solution of this equation is given in the parametric forms

$$v^2 = Cq/(q + 4)^5 \tag{4.10}$$

$$y = \tfrac{1}{6}(2q - q^2)v \tag{4.11}$$

where C is an arbitrary constant of integration. It can readily be seen that in the positive y-v quadrant (the other quadrants do not correspond to physical situations) y is a double function of v and the gradients of both branches are positive for any positive value of C. Both are acceptable solutions of the differential equation (4.5).

Thus in general there are two solutions for the angular deceleration corresponding to each angular velocity of the rod. Furthermore, there is another solution, which is given approximately, as the correction terms are small, by equation (4.4). Hence we have the untenable situation of having three possible values for the angular deceleration for each value of the angular velocity. We therefore claim that the assumption that the rod be rigid is inconsistent with the linearized version of the general theory of relativity.

5. *Gravitational Radiation from a Nonrigid Rod*

Thus the length of the rod cannot be constant if we subscribe to the linearized version of the general theory of relativity. In this section, we shall solve for the case of a nonrigid rod. Since the energy and angular momentum flux are small compared to the total energy and angular momentum of the rod, the rates of change of the angular velocity and the length of the rod are both small. Thus, to a good approximation, we put

$$\dot{\omega} = \ddot{L} = \dddot{L} = 0 \tag{5.1}$$

in the energy and angular momentum equations (3.9), (3.10). These become

$$\beta_1 + \beta_3(1 + b) = -\eta[1 + \tfrac{9}{4}\beta_1{}^2 + 6\beta_1\beta_3 + 6\beta_3{}^2 + \tfrac{9}{4}\beta_3{}^2(\beta_1 + \beta_3)^2] \tag{5.2}$$

$$\beta_1 + 2\beta_3 = -\eta[1 + \tfrac{3}{4}\beta_1{}^2 + 3\beta_1\beta_3 + 4\beta_3{}^2 + \tfrac{3}{4}\beta_3{}^3(\beta_1 + \beta_3)] \tag{5.3}$$

The last equation is a quadratic equation in β_1. To $O(\eta^3)$, we have

$$\beta_1 = -2\beta_3 - (1 + \beta_3{}^2 - \tfrac{3}{4}\beta_3{}^4)(1 - \tfrac{3}{4}\eta\beta_3{}^3)\,\eta$$
$$-\tfrac{3}{4}(1 - 22\beta_3{}^2 + 35\tfrac{1}{2}\beta_3{}^4 + 39\tfrac{3}{4}\beta_3{}^6 + 13\tfrac{5}{16}\beta_3{}^8 + \tfrac{9}{16}\beta_3^{10})\eta^3 \tag{5.4}$$

When we substitute equation (5.4) into equation (5.2), we obtain a polynomial equation in β_3

$$[2 + (93/4)\eta^2]\,\eta\beta_3{}^2 + (b - 1 + 3\eta^2)\beta_3 + (3/2)\eta^3 = 0 \tag{5.5}$$

correct to $O(\beta_3{}^2)$. We thus get

$$\beta_3 \simeq \frac{3}{2}\frac{\eta^3}{(1 - \mathrm{b})} \tag{5.6}$$

and equation (5.4) yields

$$\beta_1 \simeq -\eta - \frac{3(5 - b)}{4(1 - b)}\eta^3 \tag{5.7}$$

The expression for β_1 [equation (5.7)] differs from the usual result [equation (4.4)] by terms which are small—of the order of η^3. Furthermore, unlike the usual result, the intuition that the rate of slow-down of the rod is dependent on the nature of the material of the rod is established. From equation (5.6), we see that the rod actually increases in length, the rate of increase being proportional to η^3 and dependent on the nature of the substance. That both the rates of change of angular velocity and length be dependent on the nature of the material arises from the mutual dependence of the internal potential energy and the kinetic energy of the rod: Any increase in the length of the rod would decrease the kinetic energy and increase the potential energy of the rod.

It is easy to show from equations (5.6) and (5.7) that, approximately,

$$(\omega_0/\omega)^4 \simeq 1 + 4\eta_0\omega_0(t - t_0) \tag{5.8}$$

and

$$\left(\frac{L_0}{L}\right)^4 \simeq 1 - \frac{\eta_0{}^2}{(1 - \langle b\rangle)}\left[1 - \left(\frac{\omega}{\omega_0}\right)^6\right] \tag{5.9}$$

That is, the rod takes an infinite time to slow down and the maximum fractional increase of the rod is

$$\frac{\Delta L_{\max}}{L_0} \simeq \frac{\eta_0{}^2}{4(1 - \langle b \rangle)} \tag{5.10}$$

which is a negligibly small quantity.

6. *Discussion*

We see that, as a consequence of emission of gravitational waves from the rotating rod, the length of the rod is not constant. This conclusion comes from the existence of two independent equations—the energy and angular momentum equations, (3.9) and (3.10)—and the assumption that the mass of the rod remains unchanged. One wonders if these two equations would become identical when the flux of gravitational waves is calculated to a higher order—post-Newtonian approximation (Epstein and Wagoner 1975) or to octopole order (Bekenstein, 1973)—and the energy and angular momentum of the system evaluated in the framework of the theory of general relativity. That they would not be so follows jointly from the fact that higher-order corrections to the gravitational flux introduce terms that are at least of order c^{-2} and the fact that the rates of change of energy and angular momentum of the system depend only on $\dot{\omega}$, $\dot{L}$ and not $\ddot{\omega}$. Thus the rod will still be nonrigid though the rate of change of the length of the rod would be modified from that derived in section 5.

The contention that the length of the rod must be variable can also be seen by using the fact that the dynamical effects of the gravitational waves can be attributed entirely to a radiation reaction force acting on the system. From the Appendix, we see that there is a component of the radiation reaction force acting along the rod. This force is responsible for the change of the length of the rod.

Finally, we observe that our work is, in a sense, a converse of the result (due to Rayner, see Trautmann et al., 1964) that the angular velocity of a rigid heavy body is constant along any particle world-line in the body.

Appendix

When a system emits gravitational waves, the emission process affects the dynamics of the system since the waves carry away energy and angular momentum from it. The dynamical effects due to the gravitational waves can be attributed to an additional potential, the radiation-reaction potential ϕ_r (Misner et al., 1973). Here

$$\phi_r = \frac{G}{15c^5}\left(\frac{d^5}{dt^5} D_{\alpha\beta}\right) x^\alpha x^\beta \tag{A1}$$

Thus owing to this potential the acceleration on a particle at x_1' (measured in the rod's rotating frame directed with the x'_1 axis lying along the rod) is given by

$$a(x'_1) = -\frac{2G}{15c^5}\left[\left(x_1 \frac{d^5D_{11}}{dt^5} + x^2 \frac{d^5D_{12}}{dt^5}\right)\hat{\imath} + \left(x_1 \frac{d^5D_{12}}{dt^5} + x_2 \frac{d^5D_{22}}{dt^5}\right)\hat{\jmath}\right] \quad \text{(A2)}$$

When we substitute in the expressions for the quadrupole moment tensor from equations (2.4)-(2.7) we get

$$a(x'_1) = -\frac{GMx'_1}{15c^5}\{[(\epsilon_3 + \epsilon_1 \cos 2\theta - \epsilon_2 \sin 2\theta)\cos\theta + (\epsilon_1 \sin 2\theta + \epsilon_2 \cos 2\theta) \times \sin\theta]\hat{\imath} + [(\epsilon_1 \sin 2\theta + \epsilon_2 \cos 2\theta)\cos\theta + (\epsilon_3 - \epsilon_1 \cos 2\theta + \epsilon_2 \sin 2\theta)\sin\theta]\hat{\jmath}\} \quad \text{(A3)}$$

where

$$\epsilon_1 = \ddot{\alpha}_5 - 4\omega\dot{\alpha}_6 - 2\dot{\omega}\alpha_6 - 4\omega^2\alpha_5 \quad \text{(A4)}$$

$$\epsilon_2 = \ddot{\alpha}_6 + 4\omega\dot{\alpha}_5 + 2\dot{\omega}\alpha_5 - 4\omega^2\alpha_6 \quad \text{(A5)}$$

$$\epsilon_3 = \ddot{\alpha}_4 \quad \text{(A6)}$$

$$\alpha_4 = \tfrac{2}{3}L^2\omega^3(\beta_5 + 3\beta_3\beta_4) \quad \text{(A7)}$$

$$\alpha_5 = -2L^2\omega^3(6\beta_1 + 12\beta_3 - 3\beta_3\beta_4 - \beta_5) \quad \text{(A8)}$$

$$\alpha_6 = -8L^2\omega^3[1 - \tfrac{1}{4}(\beta_2 + 6\beta_1\beta_3 + 6\beta_3{}^2 + 6\beta_4)] \quad \text{(A9)}$$

Thus the components of the acceleration at x'_1 along the rod and perpendicular to it (measured in the direction opposite to the direction of rotation) are

$$a_{\parallel}(x'_1) = -(GM/15c^5)x'_1(\epsilon_1 + \epsilon_3) \quad \text{(A10)}$$

$$a_{\perp}(x'_1) = (GM/15c^5)x'_1\epsilon_2 \quad \text{(A11)}$$

Since $(\epsilon_1 + \epsilon_3)$ is not zero, there is a finite component of the acceleration along the rod due to the radiation-reaction potential. We see that the rod is acted on by two equal and opposite forces, F_R, each acting on half of the rod directed along the axis of the rod (in addition to the perpendicular components). From (A10), we get

$$F_R = -\frac{GM^2L(\epsilon_1 + \epsilon_3)}{60c^5} \quad \text{(A12)}$$

Consequently the length of the rod will be changed because of the existence of two equal and opposite forces directed along the rod. We note that even when the length remains unchanged, F_R is not zero.

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